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Optimizing parametric oscillators with tunable boundary conditions

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ABSTRACT

Parametric excitation or pumping is an effective method to create large oscillations by periodically altering a physical parameter of the governing dynamics. Precisely tuned pumping frequencies can lead to exponentially growing oscillations limited only by nonlinear effects like axial stretching of transversely vibrating string. It is demonstrated that a tuned passive dynamical system amplifies the otherwise limited transverse vibrations amplitudes of a nonlinear string considerably and thus increasing the selectivity of the system. This effect becomes more noticeable for shorter wavelengths where nonlinear stretching limits the obtainable vibration amplitudes severely. The present work analyses a passive dynamical system connected to one end of a taught string which parametrically couples its axial motion to transverse vibration. Analysis shows that a specific selection of parameters can reduce the limiting effect of nonlinear stretching thus allowing one to excite high-order modes with small external forces. The result can possibly affect other disciplines where effective parametric amplification is necessary.

1. Introduction

Parametric excitation [1–3], or pumping, is a well-known method to generate pure-tone, high amplitude oscillations [4]. Linear differential equations with periodically varying parameters can lead to exponentially growing amplitudes for specific combinations of frequency and amplitude of the pumping forces [3,5]. While damping only dictates the lower bound of the pumping level, the nonlinear stiffening is what limits the obtainable response levels [6] in practice. Indeed, nonlinear stiffening is an obstacle for obtaining large response levels in parametrically excited systems [7].

Although the topic of parametric excitation is well known and frequently used in practice, the most important part which deals with how the parametric excitation is realized, received little attention in the literature. Some authors have proposed static optimization by means of soft spring supports [7] but the dynamics of the boundaries is not addressed. Furthermore, the associated boundary conditions are mostly ignored or over simplified, although, as will be shown here, they play a major role in obtaining large oscillations in some cases.

The present paper opens a discussion on the importance of the boundary conditions in parametrically resonating system by analysing a simple physical system. The example shows a closed form solution for optimized dynamic boundary conditions which lead to large vibration amplitude and bypass the limitations of nonlinear stiffening.

Indeed, in the case of a taut string, axial stretching that accompanies transverse bending deflections of a constrained one-dimensional elastic structure contributes to the elevation of strain energy and thus limits the response [8]. By broadening the analysis horizon such that it incorporates the interplay between the parametrically driven system and the boundaries, a better design approach is found and a closed form solution is developed for the simple case treated here.

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Nomenclature

$w=w(x,t)$ Transverse deflection along the string (see Fig. 1)
N_0 Initial axial tension of the string.
L,A,E,ρ String's length, section area, module of elasticity and density, respectively
$q(t)$ Axial displacement of the attached mass (see Fig. 2)
$\phi(t)$ Governing mode shape of the string under parametric excitation
ω_N The natural frequency of the end-mass system at the boundary
ω_n The natural frequency of the simply supported vibrating string;s mode (nth mode).
$\Delta\delta$ Elongation or stretching of the neutral axis.

Parametric oscillations can isolate and amplify weak sinusoidal oscillations in MEMS filters, gyroscopes and electronic particle detectors [9]. Parametric resonance is often employed in low signal-to-noise conditions due to its high quality factor and superior frequency selectivity [9–12]. Tuned excitation frequencies can exploit parametric vibration to detect asymmetry and cracks in rotating shafts or quench vibrations [13,14]. There is an ever-growing research in improving the *Q*-factor of such systems [15,16] that is mostly concentrated on the parametrically vibrating elements.

In this paper, a simple elastic string is studied in order to demonstrate the effect of tuned boundary conditions in a concise manner. A similar analysis can be carried to other models, such as beam or plate for which dynamic boundary conditions can probably play a similar role.

The paper presents the coupled equations of motion that arise in the case of a parametrically resonating taut string with a spring-mass connected to it. Multiple-scale reduction is employed and exact conditions for obtaining the largest vibration amplitudes under unit force amplitude are found. The non-intuitive result is verified via numerical simulation and partially demonstrated on a dedicated experimental rig.

The paper begins in Section 2 with the description of the problem and a detailed mathematical analysis of the proposed set-up is shown. In Section 3, a numerical integration approach is employed to verify the analytical part and to illustrate the advantage of the proposed solution. Finally, some experimental results demonstrate the ability to obtain large vibration amplitudes for higher modes of vibration with high frequency selectivity and high *Q*-factor.

2. Mathematical background and equations of motion

Two similar systems undergoing parametric excitation are shown in Figs. 1 and 2. Fig. 1 shows the commonly assumed set-up and model which uses force acting directly on the string. In Fig. 2, the force is exerted on the string via a spring-mass system driven by the external force.

The heart of the matter is the amplitude limiting nonlinear stretching effect governed by the axial rigidity. Contrary to Fig. 1, the system in Fig. 2 allows the end point to move freely and therefore the total elongation (see [3]) can be computed via:

$$\Delta\delta = \int_0^L \frac{ds}{dx}\mathrm{d}x + q(t) \approx \frac{1}{2}\int_0^L w_x^2 \mathrm{d}x + q(t) \tag{1}$$

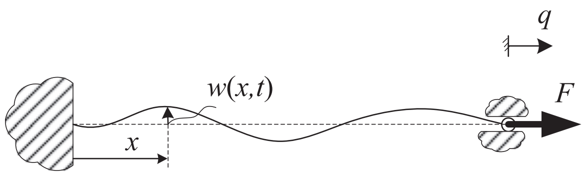

Fig. 1. Elastic string undergoing transverse vibrations due to direct modulation of tension.

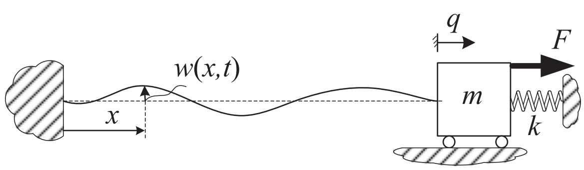

Fig. 2. Elastic string whose tension is modulated by a forced spring mass system.

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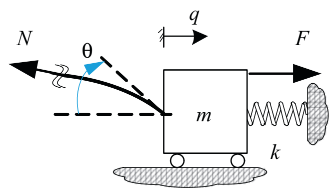

Fig. 3. Forces acting on the end mass—free body diagram.

The axial tension, including the nonlinear effect and the Initial tension—N_0, is therefore:

$$N = N_0 + \frac{EA}{L}\Delta\delta \approx N_0 + \frac{EA}{L}\left(\frac{1}{2}\int_0^L w_x^2 \mathrm{d}x + q(t)\right) \tag{2}$$

Considering the string in Fig. 2, the equation of motion becomes [17]:

$$\rho A w_{tt} = w_{xx}\left(N_0 + EA\left(\frac{1}{2L}\int_0^L (w_x^2) dx + \frac{q(t)}{L}\right)\right); \tag{3}$$

The effect of the dynamical system attached to the string is considered by analysing the free body diagram in Fig. 3. Therefore, the equation of motion of this subsystem is simply:

$$m\ddot{q} + kq = F - N\cos\theta \tag{4}$$

The projection of the tension force is obtained by approximating $\cos\theta$ with a truncated series:

$$\cos\theta \approx 1 - \frac{1}{2} w_x^2|_{x=L} \tag{5}$$

It is customary [6,8] to describe the string deflection, during parametric resonance, by a single linear mode, whose amplitude depends on a small parameter, ε, via:

$$w \approx \sqrt{\varepsilon}\phi(t)\sin\frac{n\pi x}{L} \tag{6}$$

An important point in the forgoing analysis relates the scales of transverse vibration amplitude and the axial motion at the right end. Clearly here,

$$O(q) = O\left(\int_0^L w_x^2 \mathrm{d}x\right) = O(\varepsilon).$$

Considering that the end-mass vibration is of the order of magnitude ε, and the relations in Eqs. (5) and (6), the equations of motion take the following form:

$$\varepsilon m\ddot{q} + EA\left(\frac{1}{4}\frac{\varepsilon\phi^2 n^2\pi^2}{L^2} + \frac{\varepsilon q}{L}\right) + N_0\left(1 - \frac{1}{2}\frac{\varepsilon\phi^2 n^2\pi^2}{L^2}\right) + \varepsilon kq - F = 0 \tag{7}$$

$$\rho A\ddot{\phi} + \frac{N_0\pi^2 n^2\phi}{L^2} + \frac{1}{4}\frac{\varepsilon\phi^3 EA\pi^4 n^4}{L^4} + \frac{\varepsilon\phi q EA\pi^2 n^2}{L^3} = 0 \tag{8}$$

An approximate solution can be obtained by employing the method of multiple scales. The solution can thus be expressed in terms of different time scales:

$$\begin{aligned} \phi &= \phi_0(\tau_0,\tau_1,..) + \varepsilon\phi_1(\tau_0,\tau_1,..) + \ldots; \\ q &= q_0(\tau_0,\tau_1,..) + \varepsilon q_1(\tau_0,\tau_1,..) + \ldots; \\ \tau_0 &= \tau, \tau_1 = \varepsilon\tau\ldots, \end{aligned} \tag{9}$$

The derivatives are approximated up to the order of ε^2:

$$\frac{\mathrm{d}(\cdot)}{dt} = \frac{\partial(\cdot)}{\partial\tau_0} + \varepsilon\frac{\partial(\cdot)}{\partial\tau_1} + O(\varepsilon^2), \frac{\mathrm{d}^2(\cdot)}{dt^2} = \frac{\partial^2(\cdot)}{\partial{\tau_0}^2} + 2\varepsilon\frac{\partial^2(\cdot)}{\partial\tau_0\partial\tau_1} + O(\varepsilon^2) \tag{10}$$

And the external force will be represented in terms of τ_0:

$$F = F_0 + \varepsilon F_1(\tau_0) \tag{11}$$

Substituting Eqs. (9)–(11) into Eqs. (7) and (8) and equating the coefficients of ε^0 and ε, the following equations are obtained:

$$\text{Eq. (7)}, \quad O(\varepsilon^0): N_0 - F_0 = 0 \tag{12}$$

$$\text{Eq. (7)},\quad O(\varepsilon): m\frac{\partial^2}{\partial\tau_0{}^2}q_0+kq_0+EA\left(\frac{1}{4}\frac{\phi_0^2n^2\pi^2}{L^2}+\frac{q_0}{L}\right)-\frac{1}{2}\frac{N_0\phi_0^2n^2\pi^2}{L^2}-F_1=0 \tag{13}$$

$$\text{Eq. (8)}O(\varepsilon^0): \rho A\frac{\partial^2}{\partial\tau_0{}^2}\phi_0+\frac{N_0\pi^2n^2}{L^2}\phi_0=0 \tag{14}$$

$$\text{Eq. (8)}\quad O(\varepsilon): \rho A\left(\frac{\partial^2}{\partial\tau_0{}^2}\phi_1+2\frac{\partial^2}{\partial\tau_0\partial\tau_1}\phi_0\right)+\frac{N_0\pi^2n^2}{L^2}\phi_1+\frac{1}{4}\frac{\phi_0^3EA\pi^4n^4}{L^4}+\frac{\phi_0q_0EA\pi^2n^2}{L^3}=0. \tag{15}$$

Eq. (12) is straight forward, and it equalizes the initial tension of the string to the stationary part of the external force. The solution of Eq. (14) can be written as:

$$\phi_0=\chi e^{i\omega_n\tau_0}+\overline{\chi}e^{-i\omega_n\tau_0}; \tag{16}$$

Representing the external force as having time dependency of twice the vibration frequency and the end-mass vibration as having stationary and a vibrating part, one obtains:

$$F_1=\psi e^{i2\omega_n\tau_0}+\overline{\psi}e^{-i2\omega_n\tau_0} \tag{17}$$

$$q_0=Q_0+Qe^{i2\omega_n\tau_0}+\overline{Q}e^{-i2\omega_n\tau_0} \tag{18}$$

Introducing Eqs. (17) and (18) into Eq. (13), the stationary part (Q_0) can be solved for:

$$Q_0=-\frac{1}{2}\frac{n^2\pi^2\chi\overline{\chi}(EA-2N_0)}{L(kL+EA)} \tag{19}$$

As one can expect, the mean position of the end mass, $q(t)$, represented by Q_0, is negative for a stiff string. The end-mass vibration amplitude can be found from Eq. (13) as a function of the string vibration amplitude:

$$Q=-\frac{1}{4}\frac{-4\psi L^2+\chi^2n^2\pi^2(EA-2N_0)}{L(EA+kL-4m\omega_n{}^2L)} \tag{20}$$

Substituting Eqs. (16) and (18) into Eq. (15) and eliminating secular terms [18] produces the following equation:

$$4\chi Q_0L+4\overline{\chi}QL+3\pi^2n^2\chi^2\overline{\chi}=0 \tag{21}$$

With the help of Eqs. (19)–(21), the string amplitude can be finally solved for:

$$\chi^2=\frac{-4\psi L^3A\rho(EA+kL)}{\pi^2n^2\left(-4\pi^2n^2mN_0(EA+3kL+4N_0)+3\rho LA(2N_0kL+EAkL+k^2L^2+2N_0EA)\right)} \tag{22}$$

A stiff (metallic) string is characterized by its ratio of transverse-to-axial wave velocities:

$$\frac{N_0}{EA}=\frac{c_t}{c_L}\ll 1 \tag{23}$$

Assuming low stiffness for the end-mass spring (relative to the axial string stiffness), i.e.:

$$\frac{k}{(EA/L)}\ll 1 \tag{24}$$

and under the restrictions in Eqs. (23) and (24), Eq. (22) can be approximated with:

$$\chi^2\approx\frac{-4\psi L^3A\rho}{\pi^2n^2\left(-4\pi^2n^2mN_0+3\rho LA(kL+2N_0)\right)} \tag{25}$$

An optimized system is characterized as having maximum string amplitude $|\chi|$ for constant external force amplitude $|\psi|$ by explicitly minimizing the denominator of Eq. (25). Since damping is absent from the present analysis, one seeks the solution for:

$$-4\pi^2n^2mN_0+3\rho LA(kL+2N_0)=0 \tag{26}$$

Thus, the condition for an optimal spring-mass system, in the case of a stiff string, is:

$$\frac{\omega_n^2}{\omega_N{}^2}=\frac{3}{4}; \tag{27}$$

where ω_N is the natural frequency of the free end-mass system: $\omega_N=\sqrt{\frac{k}{m}}$ and ω_n is the natural frequency of the vibrating mode: $\omega_n=\frac{\pi n}{L}\sqrt{\frac{N_0}{\rho A}}$.

It is worth noting that equating the denominator of Eq. (25) to zero does not represent infinite vibration amplitude in practice nor does it provide the exact maximum of the modal amplitude (χ). The neglected terms due to the assumptions in Eqs. (23) and (24) become significant when Eq. (26) holds. In the latter case, the amplitude is limited by the elongation of the neutral axis and the previously neglected terms are no longer small compared to Eq. (26) (which is identically 0).

3. Numerical and experimental verification

In order to verify the optimality of Eq. (27), a numerical study of Eqs. (3) and (4) via Eqs. (7) and (8) was carried out. Later, an experimental system was tested to illustrate the ability to excite higher modes with parametric excitation under the proposed set-up. The numerical analysis considers a metallic string described in Eq. (3) with the following physical properties:

$$\rho = 8\ \text{gm/cm}^3, \quad E = 200 \times 10^9\ \text{Pa}, \quad N_0 = 3.0276\ \text{N}, \quad A = 5\ \text{mm}^2, \quad L = 0.6\ \text{m}. \tag{28}$$

The model of the appended spring-mass system in Eq. (4) is slightly modified for sake of numerical stability by adding 1% damping:

$$\ddot{q} + 2\zeta\omega_N q + \omega_N^2 q = F/m - N/m\cos\theta \tag{29}$$

where m=0.73 kg, k=1001 N/m and ζ=0.01.

One can observe that

$$\omega_1 = \sqrt{\frac{N_0}{\rho A}}\frac{\pi}{L} = 7.3\ \text{Hz}, \quad \frac{\omega_1^2}{\omega_N{}^2} = 1.51 \neq 0.75 \tag{30}$$

Hence, the system does not comply with the optimally condition in Eq. (27) for mode number 1.

The time series response of the modal amplitude is shown in Fig. 4.

The tuning of the peripheral system is accomplished by alternating the end-mass as follows:

$$m = 0.365 \Rightarrow \frac{\omega_1{}^2}{\omega_N{}^2} = 0.75 \tag{31}$$

The resulting modal amplitude is described in Fig. 5.

The steady-state amplitude of the parametrically resonating string was amplified almost tenfold by connecting a tuned end-mass system.

As mentioned in the previous section, the relation in Eq. (27) is an approximation of the parameters leading to the largest modal amplitude. The numerical results described in Fig. 6 simulate Eqs. (7) and (8) represent the steady-state amplitude for different end mass values. It is shown that the steady-state amplitude reaches near maximum when the condition in Eq. (27) applies. Multiple solutions occur beyond the optimal point as can be seen from the simulated results in Fig. 6. Each dot in this plot represents the steady-state amplitude for specific end mass and slight randomly perturbed initial conditions under parametric resonance conditions of mode number 1.

One of the difficulties that arises when experimenting with parametrically resonating strings is the inability to excite parametrically high modes of vibration. The proposed procedure can be applied in order to amplify certain modes of a string including higher ones. An experimental test rig was designed in order to investigate high-order modes of vibration while applying parametric excitation. The rig is described in Fig. 7 and comprises a steel strip (#3 in Fig. 7) held at one end and connected to a horizontally moving mass (#4). The mass is supported elastically by a set of leaf springs (#5) and subjected to an external parametric excitation via a voice coil (#6). The physical dimensions are the same as the numerical ones described in Eq. (28).

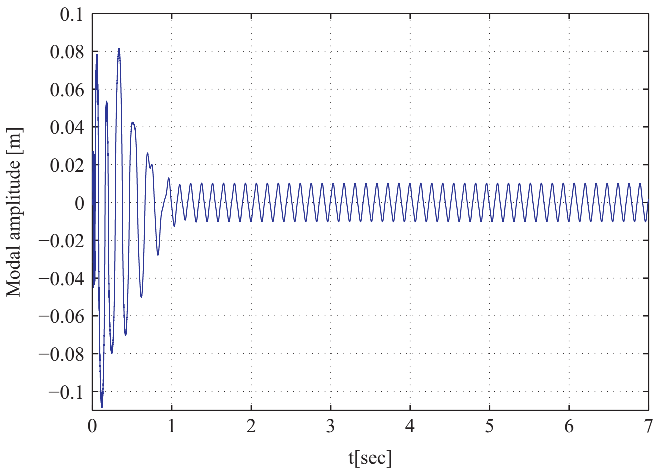

Fig. 4. Parametrically resonating string. The non-optimal case.

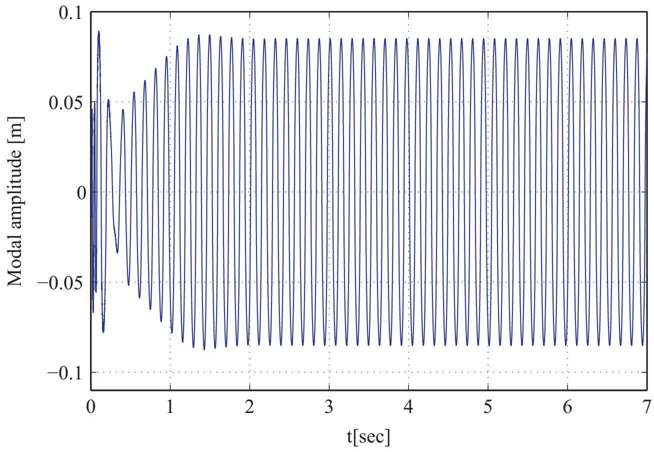

Fig. 5. Parametrically resonating string. The optimal case.

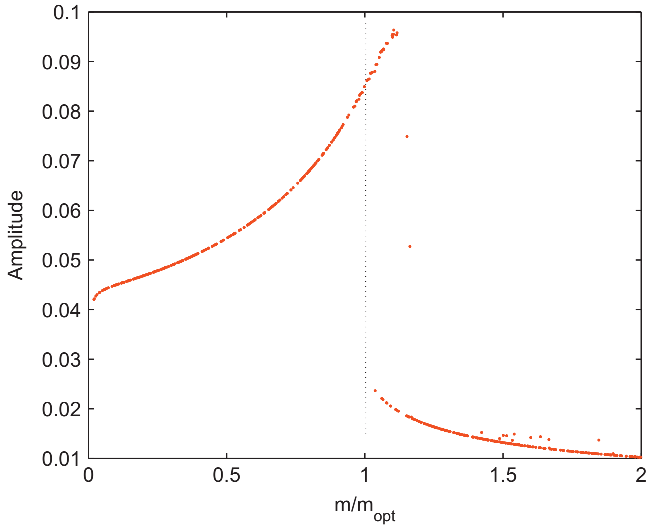

Fig. 6. Simulated response amplitude at steady state vs. end-mass value.

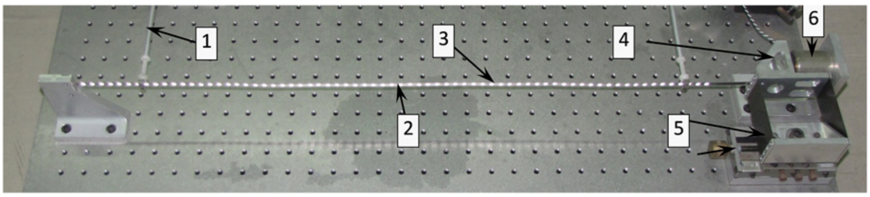

Fig. 7. Test rig of parametrically excited string. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

The response to a periodic parametric tension excitation is described in Fig. 8 as measured on the bright reflecting type points indicated #2 in Fig. 7. The colour and z-axis represent the amplitude of the parametric response while the x-axis represents the excitation frequency and the y-axis represents the location along the string. The features of high amplitude and high Q-factor are clearly recognized. These characteristics of parametric resonance are notable for all first three modes as a sharp jump in amplitude occurs at a narrow range of frequencies. The spatial shape of the individual modes can also be seen in this figure.

Eq. (27) provides the condition for an optimal appended system allowing one to choose the best combination of design parameters that maximize the string amplitudes under parametric excitation. Such a tuning condition can be of some importance as, a tool for designers or a fine tuning method for parametrically resonating systems. A similar analysis

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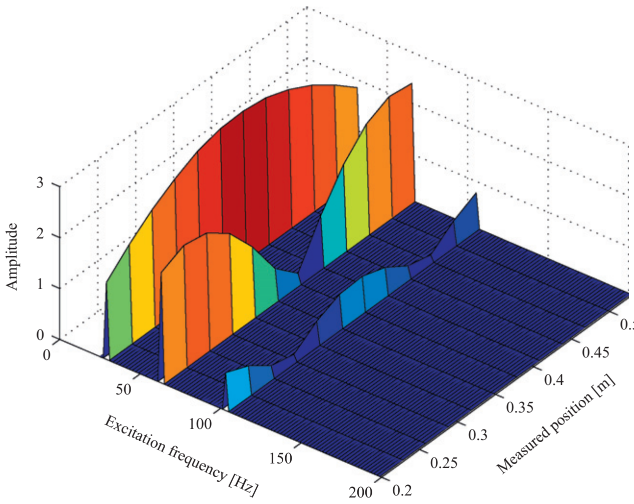

Fig. 8. String response under parametric excitation (frequency resolution 2 Hz).

considering the peripheral system, mechanical or electric, should be carried out when designing real systems for sake of obtaining the most out of the parametrically resonating systems.

4. Conclusions

Parametric excitation has clear advantages for certain applications in micro-scale and in engineering of large-scale systems. The parametric resonance is superior to ordinary resonance, in most cases, due to its high amplitudes and high selectivity. As was shown in this paper, one can improve the performance of a parametrically resonating system considerably with a relatively small design effort by tuning the boundary conditions to cooperate with the resonating system. The tuning of the peripheral elements at the boundary can increase the amplitudes of the system without force enlargement. In the presented case, a 10-fold increase has been achieved. A closed form solution linking the ratio natural frequencies between the resonating system and the boundary was found to be $\sqrt{3}/2$ and it can be achieved by the sole tuning of the boundary conditions' dynamics.

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